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Note

Hindman's Theorem and Groups

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We prove a theorem which limits the possible uncountable generalizations of Hindman's theorem.

Hindman's theorem [4] can be generalized to a theorem about countably infinite groups. This raises the question of whether some generalization of Hindman's theorem holds for uncountable groups. We have not found a solution to this last question, but assuming some form of the Generalized Continuum Hypothesis (GCH) we shall prove a negative partition relation that answers a part of this question.

First let us introduce some notation. If X is a set, we write $|X|$ for the cardinality of X. We identify a nonnegative integer with the set of preceding nonnegative integers. ω is the set of all nonnegative integers, and we identify a cardinal with the least ordinal of that cardinality.

If X is a set and κ is a cardinal (finite or infinite), then

$$[X]^\kappa = \{Y \subseteq X : |Y| = \kappa\},$$

$$[X]^{<\kappa} = \{Y \subseteq X : |Y| < \kappa\}.$$

If $A, B \in [\omega]^{<\aleph_0}$, we write $A < B$ to mean: for each $a \in A$ and $b \in B$ it is true that $a < b$.

DEFINITION 1. A sequence $\langle I_i : i \in \kappa \rangle$ of length κ, where $\kappa \leqslant \omega$, will be called *acceptable* provided

(a) $\varnothing \neq I_i \in [\omega]^{<\aleph_0}$ for each $i \in \kappa$, and

(b) $i \in j \in \kappa$ implies $I_i < I_j$.

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Using this definition, Hindman's theorem can be stated as follows:

THEOREM 2 (Hindman [4]). *For each positive integer r and each partition*

$$[\omega]^{<\aleph_0} = A_0 \cup \cdots \cup A_{r-1},$$

there is an acceptable sequence $\langle I_i : i < \omega\rangle$ *and an integer* $k \in r$ *such that*

$$\bigcup_{i \in X} I_i \in A_k$$

whenever $X \in [\omega]^{<\aleph_0}$ *with X nonempty.*

First we shall sketch a proof of Theorem 4, the generalization for groups of Hindman's theorem.

DEFINITION 3. Suppose $X = \langle x_i : i \in \omega\rangle$ is any infinite sequence of elements from a group $\langle G, \cdot\rangle$. If $\varnothing \neq I \in [\omega]^{<\aleph_0}$ we write

$$s(X, I) = x_{\rho(0)} \cdot x_{\rho(1)} \cdot \cdots \cdot x_{\rho(|I|-1)},$$

where ρ is the unique strictly increasing function $\rho\colon |I| \to I$, i.e., $s(X, I)$ is the group product of the x_i's with indices in I, with the product taken in the order of the indices. If $0 < \kappa \leqslant \omega$, we write

$$\langle X\rangle^{\kappa} = \{\langle s(X, I_i) : i \in \kappa\rangle : \langle I_i : i \in \kappa\rangle \text{ is acceptable}\}.$$

THEOREM 4. *If* $V = \langle v_i : i \in \omega\rangle$ *is an infinite sequence of distinct elements of a group* $\langle G, \cdot\rangle$, *if r is a positive integer, and if*

$$\langle V\rangle^1 = A_0 \cup A_1 \cup \cdots \cup A_{r-1},$$

then there exists an infinite sequence X of distinct elements of G and an integer $k \in r$ *with* $X \in \langle V\rangle^{\omega}$ *and*

$$\langle X\rangle^1 \subseteq A_k.$$

Simplifying notation, and assuming G is Abelian, we obtain the following corollary.

COROLLARY 5. *If* $\langle G, +\rangle$ *is an infinite Abelian group, if r is a positive integer, and if*

$$G = A_0 \cup A_1 \cup \cdots \cup A_{r-1},$$

then there exists an infinite set $X \in [G]^{\aleph_0}$. *and an integer* $k \in r$ *such that for each nonempty set* $B \in [X]^{<\aleph_0}$, *it is true that* $\Sigma B \in A_k$.

The following lemma is used in the proof of Theorem 4.

LEMMA 6. *If* $\langle G, \cdot\rangle$ *is a group and* $F \in [G]^n$ *for some* $n \in \omega$, *then for each set* $X \subseteq G$ *with* $|X| \geqslant n^2 + 1$, *there exists* $x \in X$ *such that*

$$F \cap (F \cdot x) = \varnothing.$$

Proof. Assume no such $x \in X$ exists. Then for each $x \in X$ correspond $\langle a_x, b_x\rangle \in F \times F$ such that $a_x = b_x \cdot x$. Since $|X| > |F \times F|$ there must exist $x, y \in X$ with $x \neq y$ and $\langle a_x, b_x\rangle = \langle a_y, b_y\rangle$. But then $a_x = b_x \cdot x = b_x \cdot y$, which implies $x = y$.

In order to prove Theorem 4, use Lemma 6 to find a subsequence

$$\langle W_i : i \in \omega\rangle \text{ of } \langle V_i : i \in \omega\rangle$$

such that the mapping

$$f\colon i \mapsto W_i$$

induces an injective mapping

$$F\colon [\omega]^{<\aleph_0} \to G$$

satisfying $F(I \cup I') = F(I) \cdot F(I')$ whenever $I < I'$. The theorem follows from Hindman's theorem applied to the partition

$$[\omega]^{<\aleph_0} = \hat{A}_0 \cup \hat{A}_1 \cup \cdots \cup \hat{A}_{r-1},$$

defined by:

$$I \in \hat{A}_j \qquad \text{iff } F(I) \in A_j.$$

It should be mentioned that if in the proof of Theorem 4, the use of Hindman's theorem is replaced by the use of the combination of Ramsey's theorem and Hindman's theorem to be found in [6, Theorem 2.1; 10, Lemma 2.2; 7; or 11], then the following generalization of Theorem 4 will result.

THEOREM 7. *If* $V = \langle v_i : i \in \omega\rangle$ *is an infinite sequence of distinct elements of a group* $\langle G, \cdot\rangle$, *if* r *and* n *are positive integers, and if*

$$\langle V\rangle^n = A_0 \cup A_1 \cup \cdots \cup A_{r-1},$$

then there exists an integer $k \in r$ *and an infinite sequence* X *of distinct elements of* G *with* $X \in \langle V\rangle^\omega$ *and*

$$\langle X\rangle^n \subseteq A_k.$$

Of course there is a finitary version of Theorem 7. Also, Theorem 7 can be further generalized along the lines of the combination of Silver's partition theorem and Hindman's theorem to be found in [11, 6, or 7].

QUESTION 8. *Is it true that for each infinite group $\langle G, \cdot\rangle$ and each partition $G = A_0 \cup A_1$, there must exist $k \in 2$ and a set $X \in [G]^{\aleph_0}$ such that for each $1 \leqslant n < \omega$ and each finite sequence $\langle x_i : i \in n\rangle$ of distinct elements from X, we have $x_0 \cdot x_1 \cdot \cdots \cdot x_{n-1} \in A_k$? Theorem 4 says that this is true if we restrict our attention to products formed in a single predetermined order, but what happens if all products are considered?*

A finitary version of the above question is not meaningul, since the following result is known: To each $n \in \omega$ there corresponds $N \in \omega$ such that if G is a finite group with $|G| \geqslant N$, then G has an Abelian subgroup of cardinality at least n. (See [5].)

We suspect that the answer to Question 8 is negative. But a counterexample would require an infinite group without any infinite Abelian subgroups, and so one would probably seek a suitable partition of the Novikov–Adjan groups [8].

We turn to the primary result of this paper, Theorem 9, which limits the possible uncountable generalizations of Theorem 4 and Corollary 5.

THEOREM 9. *If κ is a cardinal, $\kappa \geqslant \aleph_0$, if $2^\kappa = \kappa^+$, and if $\langle G, \cdot\rangle$ is a group with $|G| = \kappa^+$, then there exists a function $f\colon G \to \kappa^+$ such that for each set $X \in [G]^{\kappa^+}$ there exists an element $a \in X$ with the property that for each $\alpha \in \kappa^+$ there corresponds $b \in X$ with $f(a \cdot b) = \alpha$.*

We would like to thank Fred Galvin for pointing out the above strengthened version of our original theorem.

Proof of Theorem 9. Write G as a union of subgroups, say

$$G = \bigcup_{\mu \in \kappa^+} G_\mu$$

where

(1) $|G_\mu| = \kappa$ for each $\mu \in \kappa^+$,

(2) $G_\mu \subseteq G_{\mu+1}$ and $|G_{\mu+1} - G_\mu| = \kappa$, and

(3) $G_\mu = \bigcup_{\nu \in \mu} G_\nu$ if μ is a limit ordinal.

First, enumerate G as $G = \{b_\alpha : \alpha \in \kappa^+\}$. Then proceed by induction. Let G_0 be the subgroup of G generated by $Q_0 = \{b_\alpha : \alpha \in \kappa\}$, so $|G_0| = \kappa$. Next, suppose $\mu \in \kappa^+$, and suppose G_ν has been chosen for each $\nu \in \mu$. If μ is limit, (3) defines G_μ. If $\mu = \nu + 1$, let $\alpha(\mu)$ be the least $\alpha \in \kappa^+$ such that $b_\alpha \in G - G_\nu$. Then let G_μ be the subgroup of G generated by $\{b_{\alpha(\mu)}\} \cup G_\nu$. It is easy to check that this construction assures that the subgroups G_μ ($\mu \in \kappa^+$) have the desired properties.

Next, enumerate the collection of all pairs $\langle A, g\rangle$ such that $A \in [G]^{\kappa}$ and g is a function, $g\colon A \to \kappa^{+}$, as

$$\{\langle A_{\alpha}, g_{\alpha}\rangle \colon \alpha \in \kappa^{+}\}.$$

(This is where the assumption $2^{\kappa} = \kappa^{+}$ is used.) Moreover, we require this enumeration to satisfy: $A_0 \subseteq G_0$.

Now we proceed with an inductive definition of the function $f\colon G \to \kappa^{+}$. First set $f(x) = 0$ for each $x \in G_0$. Then at step μ, if $f(x)$ is defined for each $x \in \bigcup_{\nu\leqslant\mu} G_{\nu}$, we want to define $f(x)$ for all

$$x \in G_{\mu+1} - \left(\bigcup_{\nu\leqslant\mu} G_{\nu}\right).$$

To do this, first enumerate

$$\left(G_{\mu+1} - \bigcup_{\nu\leqslant\mu} G_{\nu}\right) \times \left\{\langle A\alpha, g_{\alpha}\rangle \colon \alpha \leqslant \mu \text{ and } A_{\alpha} \subseteq \bigcup_{\nu\leqslant\mu} G_{\nu}\right\},$$

as

$$\{\langle x_{\gamma}, A_{\alpha(\gamma)}, g_{\alpha(\gamma)}\rangle \colon \gamma \in \kappa\}. \tag{*}$$

(The requirement $A_0 \subseteq G_0$ implies this set is nonempty.) Then induct on $\gamma \in \kappa$. At the γth step, consider the set

$$A_{\alpha(\gamma)} \cdot x_{\gamma} \subseteq \left(G_{\mu+1} - \bigcup_{\nu\leqslant\mu} G_{\nu}\right).$$

Since $| A_{\alpha(\gamma)} \cdot x_{\gamma} | = \kappa$, we can pick an element $a_{\gamma} \in A_{\alpha(\gamma)}$ such that $f(a_{\gamma} \cdot x_{\gamma})$ has not yet been defined. Set $f(a_{\gamma} \cdot x_{\gamma}) = g_{\alpha(\gamma)}(a_{\gamma})$. After this has been done κ times, i.e., for all $\gamma \in \kappa$, if $f(x)$ remains undefined for $x \in G_{\mu+1} - \bigcup_{\nu\leqslant\mu} G_{\nu}$, then set $f(x) = 0$.

This double induction completes the definition of $f\colon G \to \kappa^{+}$. It remains to show that f has the desired property, i.e., we must show that to each $X \in [G]^{\kappa^{+}}$ there corresponds $a \in X$ with the property that for each $\alpha \in \kappa^{+}$ there exists $b \in X$ with $f(a \cdot b) = \alpha$.

Suppose this is false, i.e., there exists $X \in [G]^{\kappa^{+}}$ such that for each $a \in X$ there is an $h(\alpha) \in \kappa^{+}$ such that for all $b \in X$, $f(a \cdot b) \neq h(\alpha)$. Pick $A \in [X]^{\kappa}$, and let $g\colon A \to \kappa^{+}$ be the restriction of the function h to the set A. Now $A \in [G]^{\kappa}$ and $g\colon A \to \kappa^{+}$, so the pair $\langle A, g\rangle$ must appear as some $\langle A_{\alpha}, g_{\alpha}\rangle$ in the enumeration of all such pairs in our construction of f. Suppose $\langle A, g\rangle = \langle A_{\alpha'}, g_{\alpha'}\rangle$.

Now $|X| = \kappa^+$ requires that there exist some $\mu' > \alpha'$ and some $x \in X$ with

$$x \in G_{\mu'+1} - \bigcup_{\nu \leqslant \mu'} G_\nu ,$$

and we may also take μ' so large that $A \subseteq G_{\mu'}$. Then $\langle x, A, g \rangle$ would have appeared in the enumeration (*) during the definition of f on $G_{\mu'+1} - \bigcup_{\nu \leqslant \mu'} G_\nu$. So we know that some element $a \in A$ was selected with $f(a \cdot x) = g(a) = h(a)$. But the definition of $h(a)$ required that for all $b \in X$, $f(a \cdot b) \neq h(a)$. This contradiction proves the theorem.

COROLLARY 10. *If κ is an infinite cardinal with $2^\kappa = \kappa^+$, and if X is a set with $|X| = \kappa^+$, then there is a partition*

$$[X]^{\leqslant \kappa} = \bigcup_{\alpha \in \kappa^+} A_\alpha$$

such that given any family $\mathscr{F} \subseteq [X]^{\leqslant \kappa}$ of pairwise disjoint subsets of X with $|\mathscr{F}| = \kappa^+$, and given any $\beta \in \kappa^+$, there must exist sets $F, G \in \mathscr{F}$ with

$$F \cup G \in A_\beta .$$

This is an immediate application of the theorem with $G = [X]^{\leqslant \kappa}$, the group product being symmetric difference, and with $A_\alpha = f^{-1}(\alpha)$.

COROLLARY 11. *Assuming the continuum hypothesis, and writing* **R** *for the set of all real numbers, there exists a partition*

$$\mathbf{R} = A_0 \cup A_1$$

such that to each $X \in [\mathbf{R}]^{\aleph_1}$ and each $i \in 2$ there correspond real numbers $a, b \in X$ with $a + b \in A_i$.

This answers a question raised by Erdös in [1]. We have learned that Erdös has (and possibly others have) also solved this problem.

Also, Theorem 9 strengthens the classic proof (see Erdös *et al.* [3]) that if $2^\kappa = \kappa^+$, then

$$\kappa^+ \not\rightarrow [\kappa^+]^2_{\kappa^+} .$$

Furthermore, if we take f in Theorem 9 so $f: G \to G$ (instead of $f: G \to \kappa^+$), then considering f as an additional, unary operation on G, we see that $\langle G, \cdot, f \rangle$ is a Jònnson algebra of cardinality κ^+. So we have a new proof of the existence of Jònnson algebras with cardinality κ^+ assuming $2^\kappa = \kappa^+$. (Erdös and Hajnal [2] gave the first proof of the existence of Jònnson algebras on κ^+ if $2^\kappa = \kappa^+$.)

REFERENCES

1. P. ERDÖS, Problems and results in combinatorial analysis, *in* "Combinatorics," Proc. Symp. Pure Math., Vol. 19, pp. 77–89, Amer. Math. Soc., Providence, R.I., 1971.
2. P. ERDÖS AND A. HAJNAL, On a problem of B. Jónnson, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys.* **14** (1966), 61–99.
3. P. ERDÖS, A. HAJNAL, AND R. RADO, Partition relations for cardinals, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 93–196.
4. N. HINDMAN, Finite sums from sequences within cells of a partition of *N*, *J. Combinatorial Theory Ser. A*, **17** (1974), 1–11.
5. B. HUPPERT, "Endliche Gruppen I," Die Grundlehren der Math. Wissenschaften, Band 134, Springer-Verlag, New York, 1967.
6. K. R. MILLIKEN, Ramsey's theorem with sums on unions, *J. Combinatorial Theory Ser. A* **18** (1975), 276–289.
7. K. R. MILLIKEN, "Some Results in Ramsey Theory," Doctoral dissertation, U.C.L.A., 1975.
8. P. S. NOVIKOV AND S. I. ADJAN, Infinite periodic groups I, II and III, *Math. USSR Izv.* **2** (1968), 209–236, 241–479, and 665–685.
9. J. SILVER, Every analytic set is Ramsey, *J. Symbolic Logic* **35** (1970), 60–64.
10. A. TAYLOR, A canonical partition relation for finite subsets of ω, *J. Combinatorial Theory Ser. A*, to appear.
11. A. TAYLOR, "Some Results in Partition Theory," Doctoral dissertation, Dartmouth College, 1975.